

Course notes

Every topic filed so far, by subject rather than by lesson: key ideas, load-bearing formulas, one worked example per topic and short practice with answers. No dates, weights or policies – for those see the course page.

ECE 207 asks one question six times: given a system and an input, what comes out? Chapter 1 settles which systems the question applies to; the five chapters after answer it in a different setting, generated by two choices — continuous vs. discrete time, and time domain vs. frequency domain. This document is organised the same way: Chapters 1 and 2 in the depth the course has reached so far, then one topic per remaining chapter at preview depth until a lesson of its own is built.

Signals and systems

Which systems the rest of the course’s machinery applies to, and the six topics Chapter 1 breaks that question into.

A signal is continuous-time, $x(t)$, when time ranges over the reals, or discrete-time, $x[n]$, when it ranges over the integers. It is periodic when $x(t + T) = x(t)$ for every t , with the smallest such T the fundamental period.

Periodic discrete-time signals

Sampling at the integers can destroy periodicity: $\cos(3t)$ repeats every $2\pi/3$, but $\cos(3k)$ never repeats. $x[k]$ is periodic (Def 1.1.12) when $x[k + N] = x[k]$ for every integer k and some positive integer N ; the smallest such N is the fundamental period. $\cos(\Omega_0 k)$ repeats after N samples exactly when $\Omega_0 N = 2\pi m$ for an integer m , so:

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$$\cos(\Omega_0 k) \text{ is periodic} \iff \frac{\Omega_0}{2\pi} = \frac{m}{N} \text{ is rational; } N = \text{the denominator in lowest terms.}$$

The period is not $2\pi/\Omega_0$: for $\cos(3\pi k/5)$, $\Omega_0/2\pi = 3/10$, so $N = 10$ ($m = 3$ turns fit in 10 samples), while $2\pi/\Omega_0 = 10/3$ is not even an integer. A sum of periodic sequences is always

periodic (Fact 1.1.14), with period $\text{lcm}(N_1, N_2)$ — unlike continuous time, where the periods need a rational ratio. Run the rational test on every term first: one non-periodic term makes the whole sum non-periodic.

WORKED FOR THIS LESSON · EASY · AI, FROM THE CHAPTER 1 NOTES

Find the fundamental period of $\cos(\pi k/6) + \sin(2\pi k/9)$, and say whether $\cos(\pi k/10) + \sin(k/4)$ is periodic.

DERIVATION · SKIP ON RE-READ

First term: $\Omega_0/2\pi = 1/12$, so $N_1 = 12$. Second: $(2\pi/9)/2\pi = 1/9$, so $N_2 = 9$. The sum has $N = \text{lcm}(12, 9) = 36$.

In the second signal, $\sin(k/4)$ has $\Omega_0/2\pi = 1/(8\pi)$, which is irrational, so that term never repeats and neither does the sum.

Transformations of the time variable

Every transformation changes the clock, not the signal: the value at time t in the new signal is a value x already had, at some other time.

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Inversion: $y(t) \triangleq x(-t)$, Scaling ($a > 0$): $y(t) \triangleq x(at)$, Shift: $y(t) \triangleq x(t - t_0)$.

Scaling with $a = 2$ compresses toward the origin, $a = 1/2$ stretches; a shift by $t_0 > 0$ moves the picture right. In discrete time a and k_0 must be integers.

WORKED FOR THIS LESSON · MEDIUM · AI, FROM THE CHAPTER 1 NOTES

The Example 1.1.28 triangle lives on $0 \leq t < 3$ with its peak at $t = 2$. Where is $y(t) = x(-2t + 4)$ nonzero, where is its peak, and which side is steep?

DERIVATION · SKIP ON RE-READ

Nonzero where $0 \leq -2t + 4 < 3$. Subtract 4 and divide by -2 , which flips both inequalities: $1/2 < t \leq 2$.

The peak is where $-2t + 4 = 2$, so at $t = 1$. The rising side of x (argument 0 to 2) maps to t from 2 down to 1, a length of 1. The falling side (argument 2 to 3) maps to t from 1 down to $1/2$, a length of $1/2$. The steep side is on the left.

Energy and power

Two ways to measure the size of a signal, and why a signal has at most one of them.

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$$E \triangleq \int_{-\infty}^{\infty} |x(t)|^2 dt, \quad P \triangleq \lim_{a \rightarrow \infty} \frac{1}{2a} \int_{-a}^a |x(t)|^2 dt, \quad P = \frac{1}{T} \int_0^T |x(t)|^2 dt \text{ (periodic)}.$$

E finite $\Rightarrow$ *energy* signal; $0 < P < \infty \Rightarrow$ *power* signal. If E is finite then $P = 0$; if P is finite and nonzero then $E = \infty$ – a signal is never both. An everlasting sinusoid has infinite energy and power half its amplitude squared.

Even and odd parts

Every signal is an even signal plus an odd one, and there is a formula for each. x is even when $x(-t) = x(t)$, odd when $x(-t) = -x(t)$; an odd signal passes through zero, since $x(0) = -x(0)$.

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$$x_e(t) \triangleq \frac{1}{2} [x(t) + x(-t)], \quad x_o(t) \triangleq \frac{1}{2} [x(t) - x(-t)].$$

System properties

Linear. Response to $a_1x_1 + a_2x_2$ is $a_1y_1 + a_2y_2$; a zero input must give a zero output. **Time invariant** (Def 1.2.10):

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$$x_2(t) \triangleq x_1(t - t_0) \implies y_2(t) = y_1(t - t_0), \quad \text{for every } t_0.$$

Any t or k appearing *outside* the input's argument breaks time invariance. **Memoryless** (the prof's definition): $y(t_0)$ depends only on the input value at that same time t_0 , not on $x(t)$ at any other t ; so $y = x^2$, $y[k] = kx[k]$ and $v = Ri$ are memoryless, and a pure delay is not. **Causal** (Def 1.2.16): if two inputs agree for all $t \leq t_0$, their outputs must agree at t_0 – the output never reads the future. Memoryless implies causal; the converse fails, since the RC circuit is causal and has memory. LTI (linear and time invariant) is the price of admission for everything after this chapter.

TABLE 1 *four systems against all four properties*

SYSTEM	LINEAR	TI	MEMORYLESS	CAUSAL
$y = 1 + x$ (Ex 1.2.8)	no	yes	yes	yes
RC circuit (Ex 1.2.9, 1.2.14, 1.2.18)	yes	yes	no	yes
$y = x^2$ (Ex 1.2.11)	no	yes	yes	yes
$y[k] = k x[k]$ (Ex 1.2.12)	yes	no	yes	yes
$y[k] = \sum_{j=0}^4 x[k+j]$ (Ex 1.2.19)	yes	yes	no	no
$y(t) = \int_{-\infty}^t \tau x(\tau) d\tau$ (Tut. 2 Q6)	yes	no	no	yes

WORKED FOR THIS LESSON • MEDIUM • AI, FROM THE CHAPTER 1 NOTES

Classify $y(t) = x(t-1)$, $y(t) = t x(t)$ and $y(t) = x(2t)$ on all four properties.

DERIVATION • SKIP ON RE-READ

$x(t-1)$: linear, time invariant, causal (it looks one second back), not memoryless.

$t x(t)$: linear, memoryless, causal, but time varying: $y_2(t) = t x_1(t-t_0)$ while $y_1(t-t_0) = (t-t_0) x_1(t-t_0)$. (This is also why $y(t) = t x(t)$ has no single, shiftable impulse response: without time invariance the convolution machinery of the next section never gets started.)

$x(2t)$: linear, but time varying ($y_2(t) = x_1(2t-t_0)$ while $y_1(t-t_0) = x_1(2t-2t_0)$), not memoryless, and not causal: $y(1) = x(2)$ reads the future.

TUTORIAL 2, Q5(A) AND Q6 • MEDIUM • OFFICIAL

Is $\dot{y} + y = \cos x(t)$, $y(0) = 0$, linear? Is $y(t) = \int_{-\infty}^t \tau x(\tau) d\tau$ time invariant?

DERIVATION • SKIP ON RE-READ

Take $x_1 = a$ and $x_2 = -a$ with $a \neq 0$. Both give $\dot{y} + y = \cos a$ from rest, so $y_2 = y_1 = \cos a (1 - e^{-t})$; a linear system would need $y_2 = -y_1$. Not linear. (Zero input already fails: it gives $1 - e^{-t} \neq 0$.)

For $x_2(t) = x_1(t-t_0)$, substitute $\sigma = \tau - t_0$:

$$y_2(t) = \int_{-\infty}^{t-t_0} (\sigma + t_0) x_1(\sigma) d\sigma = y_1(t-t_0) + t_0 \int_{-\infty}^{t-t_0} x_1(\sigma) d\sigma.$$

The extra term is nonzero for a general input, so it is not time invariant: the τ outside the input's argument is the giveaway.

TUTORIAL 2, Q8 · MEDIUM · OFFICIAL

A linear system answers $u(t)$ with $y = t$ and $u(t - 1)$ with $y = t/2$. Is it causal? Time invariant? What is its response to the input that is 0 for $t \leq 0$, 2 on $(0, 1]$ and 1 for $t > 1$?

DERIVATION · SKIP ON RE-READ

Not causal: both inputs are zero for $t < 0$, yet both outputs are nonzero there. Not time invariant: a shift would give $t - 1$ for the delayed step, not $t/2$. The input is $2u(t) - u(t - 1)$, and linearity alone (no shifting allowed) gives $y = 2t - t/2 = \frac{3}{2}t$.

LTI response by shifted copies

Know the response to one pulse, and you know the response to any staircase built from it. If y_0 is the response to x_0 , then by linearity and time invariance:

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$$x(t) = \sum_i c_i x_0(t - t_i) \implies y(t) = \sum_i c_i y_0(t - t_i).$$

Write x as scaled, delayed copies of x_0 ; the output side is copy-and-paste.

TUTORIAL 1, Q7 · MEDIUM · OFFICIAL

An LTI system responds to the unit pulse $x_0(t)$ (1 on $(0, 1]$, 0 elsewhere) with $y_0(t)$. Find its response to the input that is 1 on $(0, 1]$, 3 on $(1, 2]$, 0 on $(2, 3]$, -2 on $(3, 4]$ and 0 elsewhere.

DERIVATION · SKIP ON RE-READ

Each unit interval is a copy of x_0 delayed to start there, scaled by the height; the gap on $(2, 3]$ has coefficient 0, so it does not appear:

$$x(t) = x_0(t) + 3x_0(t - 1) - 2x_0(t - 3), \quad y(t) = y_0(t) + 3y_0(t - 1) - 2y_0(t - 3).$$

Network models with $D = d/dt$

Turn a circuit into one differential equation relating input and output. A series RLC circuit driven by $x(t)$ with loop current $y(t)$: KVL adds the three element voltages, then differentiating once removes the integral:

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$$L \frac{d^2 y}{dt^2} + R \frac{dy}{dt} + \frac{1}{C} y(t) = \frac{dx}{dt}.$$

With more than one loop, write $D \triangleq d/dt$, treat it as a number, eliminate currents by Cramer's rule, then read each power of D back as a derivative.

EXAMPLE 1.2.22 · MEDIUM · OFFICIAL

Loop 1 has a 2Ω resistor and a 2 H inductor, loop 2 has a 4Ω resistor and a 1 H inductor, sharing a 3Ω resistor carrying $y_1 - y_2$. Input $x(t)$ drives loop 1; output $y = y_1$. Find the differential equation.

DERIVATION · SKIP ON RE-READ

$$(2D + 5)y_1 - 3y_2 = x(t), \quad -3y_1 + (D + 7)y_2 = 0.$$

Cramer's rule for y_1 :

$$y_1 = \frac{(D + 7)x}{(2D + 5)(D + 7) - 9} = \frac{(D + 7)x}{2D^2 + 19D + 26}.$$

Read the powers of D back as derivatives:

$$2 \frac{d^2 y}{dt^2} + 19 \frac{dy}{dt} + 26 y(t) = \frac{dx}{dt} + 7 x(t).$$

Node equations do the same job with KCL: sum the currents leaving each node using $i = v/R$, $i = C dv/dt$ and $v = L di/dt$, then eliminate every variable but the output.

TUTORIAL 3, Q1 · MEDIUM · OFFICIAL

A current source i_s feeds node A, which has R_1 to ground; an inductor L runs from A to node B, which has C in parallel with R_2 to ground. Find the equation for the voltage v across R_2 .

DERIVATION · SKIP ON RE-READ

KCL at A: $i_s = v_1/R_1 + i_L$. KCL at B: $i_L = v/R_2 + C v'$. The inductor: $v_1 - v = L i_L'$. Substitute i_L from B into the inductor equation for v_1 , then both into A:

$$LC \frac{d^2 v}{dt^2} + \left(\frac{L}{R_2} + R_1 C \right) \frac{dv}{dt} + \left(1 + \frac{R_1}{R_2} \right) v = R_1 i_s.$$

Check with $L = C = 0$: $v = R_1 R_2 i_s / (R_1 + R_2)$, the current divider. (The posted solution drops the C on v'' in one intermediate line; its final line agrees with this.)

With an ideal op-amp, the inverting node is at 0 V and draws no current. In Tutorial 3 Q3, feedback through $R_2 \parallel L$ gives $\frac{L}{R_2} \frac{di_L}{dt} + i_L = -i_s$; the resistor R_1 in series with the ideal current source does not appear.

Practice, with answers

- (Worked for this lesson, easy, ai, from the Chapter 1 notes)** Find the fundamental period of $\sin(7\pi k/12)$ and of $\cos(\pi k/10) + \sin(3\pi k/4)$.
Answer: $7/24$ is in lowest terms, so $N = 24$; the sum has $N = \text{lcm}(20, 8) = 40$.
- (Tutorial 1, Q1(b), easy, official)** Find the energy of $x(t) = t[u(t) - u(t - 5)]$.
Answer: $E = \int_0^5 t^2 dt = 125/3$.
- (Worked for this lesson, easy, ai, from Tutorial 1)** Classify $x(t) = e^{-2t}u(t)$ and $x(t) = u(t)$ as energy or power signals, and give the number.
Answer: $e^{-2t}u(t)$ is an energy signal, $E = 1/4$ (power 0); $u(t)$ is a power signal, $P = 1/2$ (energy infinite).
- (Worked for this lesson, medium, ai, from the Chapter 1 notes)** Find the even and odd parts of $x(t) = e^{-t}u(t)$ for $t \neq 0$.
Answer: $x_e(t) = \frac{1}{2}e^{-|t|}$, $x_o(t) = \frac{1}{2}\text{sgn}(t)e^{-|t|}$.
- (Worked for this lesson, easy, ai, from Tutorial 1)** Same LTI system as above. The input is 2 on $(0, 2]$, -1 on $(2, 3]$ and 0 elsewhere. Write the output.
Answer: $x(t) = 2x_0(t) + 2x_0(t - 1) - x_0(t - 2)$, so $y(t) = 2y_0(t) + 2y_0(t - 1) - y_0(t - 2)$.
- (Worked for this lesson, medium, ai, from the Chapter 1 notes)** Change the 4Ω resistor in loop 2 of Example 1.2.22 to 1Ω , keep everything else, and find the differential equation for $y = y_1$.
Answer: $-3y_1 + (D+4)y_2 = 0$ gives $y_1 = (D+4)x/(2D^2+13D+11)$, so $2y'' + 13y' + 11y = x' + 4x$.

Time-domain analysis

Why the response to one sharp kick tells you the response to everything.

The unit impulse

The impulse is the limit of a pulse that keeps its area while it narrows. Let u_ϵ rise linearly from 0 at $t = 0$ to 1 at $t = \epsilon$ and stay there; its derivative δ_ϵ is a block of height $1/\epsilon$ on $(0, \epsilon)$, area 1, and

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$$\delta(t) \triangleq \lim_{\epsilon \rightarrow 0} \delta_\epsilon(t), \quad \int_{-\infty}^{\infty} \delta(t) dt = 1, \quad \delta(t) = 0 \ (t \neq 0), \quad u(t) = \int_{-\infty}^t \delta(\tau) d\tau, \quad \frac{du}{dt} = \delta.$$

$\delta(0)$ is never evaluated: $a \delta(t)$ is an arrow of area a , and the impulse only means something inside an integral.

Sifting

An impulse inside an integral picks out the value of everything else at the spike (Theorem 2.1.1, Remark 2.1.2):

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$$x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau = \int_{-\infty}^{\infty} x(t - \tau) \delta(\tau) d\tau, \quad \int_{-\infty}^{\infty} x(t) \delta(t - a) dt = x(a).$$

The reason: $\delta_\epsilon(t - \tau)$ is a block of height $1/\epsilon$ on $[t - \epsilon, t]$, so the integral is the average of x over that strip, which tends to $x(t)$. The spike must lie inside the limits: $\int_0^1 \tau \delta(\tau - 3) d\tau = 0$, while $\int (\tau^2 + 1) \delta(\tau - 2) d\tau = 5$.

Higher-order impulses $\delta^{(n)} = d^n \delta / dt^n$ sift out derivatives (Theorem 2.1.3, by integration by parts):

$$\int_{-\infty}^{\infty} x(\tau) \delta^{(n)}(t - \tau) d\tau = x^{(n)}(t),$$

so $\int \tau^3 \delta^{(1)}(t - \tau) d\tau = 3t^2$, and a system whose impulse response is $\delta^{(1)}$ is a differentiator.

Impulse response and convolution

A system is *initially at rest* (Def 2.2.1) when an input that is zero up to t_0 gives an output that is zero up to t_0 . Its *impulse response* $h(t)$ (Def 2.2.2) is its output for the input $\delta(t)$. Write x as a staircase of narrow pulses, $x \approx \sum_k x(k\epsilon) \delta_\epsilon(t - k\epsilon)$; time invariance shifts the response

to each pulse, linearity adds them, and letting $\epsilon \rightarrow 0$ gives Theorem 2.2.8: for an LTI system initially at rest,

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$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau = (x * h)(t) = (h * x)(t).$$

Read it as: the input near τ is an impulse of area $x(\tau) d\tau$ that launches a copy of h at τ , and by time t that copy has run for $t - \tau$. So h is the complete description of an LTI system.

The system is causal exactly when $h(t) = 0$ for $t < 0$ (the zero input and δ agree up to any $t_0 < 0$, so their outputs 0 and h must too). Then

$$y(t) = \int_{-\infty}^t x(\tau) h(t - \tau) d\tau, \quad \text{or} \quad \int_0^t \text{ when the input also starts at 0.}$$

So $e^{-2t}u(t)$ is causal, $u(t+1) - u(t-1)$ is not ($h(-0.5) = 1$), and $h = \delta(t-3)$ is a pure delay, $y = x(t-3)$.

WORKED FOR THIS LESSON · EASY · AI, FROM THE CHAPTER 2 NOTES

$h(t) = e^{-2t}u(t)$ and $x(t) = u(t)$. Find y .

DERIVATION · SKIP ON RE-READ

Both are causal, so for $t \geq 0$

$$y(t) = \int_0^t e^{-2(t-\tau)} d\tau = \frac{1 - e^{-2t}}{2},$$

and $y = 0$ for $t < 0$. Check: $y(0) = 0$, and $y \rightarrow 1/2$, the area of h — convolving with u is a running integral.

Convolution by flip and shift

Rename the variable τ , flip one signal (the shorter is easier) and slide it: $x(t - \tau)$ is the mirror image of x moved t to the right, so each of its edges sits at t plus where it sat at $t = 0$. A breakpoint in t is where a moving edge passes a fixed edge. Between breakpoints the overlap has one form; integrate the product over it, with limits that may depend on t , and write the answer piecewise — never add the pieces. Whether a join uses $<$ or $\leq$ does not matter.

TUTORIAL 2 Q5 AND TUTORIAL 3 Q5 · MEDIUM · OFFICIAL

$x(t) = 2 - t$ on $[0, 2]$ and 0 elsewhere. Find $(x * u)(t)$.

DERIVATION · SKIP ON RE-READ

Flip and slide the step: $u(t - \tau)$ is 1 for $\tau \leq t$. Its edge passes x 's edges at $t = 0$ and $t = 2$.

$$(x * u)(t) = \begin{cases} 0, & t \leq 0 \quad (\text{no overlap}), \\ \int_0^t (2 - \tau) d\tau = 2t - t^2/2, & 0 < t \leq 2, \\ \int_0^2 (2 - \tau) d\tau = 2, & t > 2. \end{cases}$$

The pieces meet at $t = 2$ ($4 - 2 = 2$), and the answer levels off at the area of the ramp, as a running integral should.

Convolution by indicator functions

The same breakpoints, found by algebra instead of a sketch. Write $I[\alpha, \beta](\tau)$ for 1 on $\alpha \leq \tau \leq \beta$ and 0 elsewhere (0 everywhere when $\alpha > \beta$). Then

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$$I[\alpha_1, \beta_1] I[\alpha_2, \beta_2] = I[\max(\alpha_1, \alpha_2), \min(\beta_1, \beta_2)], \quad \int x I[\alpha, \beta] d\tau = \begin{cases} \int_{\alpha}^{\beta} x d\tau, & \alpha \leq \beta, \\ 0, & \alpha > \beta. \end{cases}$$

Write each factor as a formula times an indicator in τ ; for the shifted one, solve its support for τ (e.g. $-5 \leq t - \tau \leq 10 \iff t - 10 \leq \tau \leq t + 5$). The cases are where each argument wins the max or the min, plus the range with $\alpha > \beta$, where the answer is 0. The output runs from the sum of the two left ends to the sum of the two right ends.

TUTORIAL 3, Q6 · MEDIUM · OFFICIAL

An LTI system at rest has $h = 1$ on $[0, 1]$ and 0 elsewhere. Find its response to $x(t) = \sin(\pi t) u(t)$.

DERIVATION · SKIP ON RE-READ

$x(\tau) = \sin(\pi\tau) I[0, \infty)$ and $h(t - \tau) = I[t - 1, t]$, so $\alpha = \max(0, t - 1)$ and $\beta = t$.

$$y(t) = \begin{cases} 0, & t < 0 \quad (\alpha > \beta), \\ \int_0^t \sin \pi \tau d\tau = (1 - \cos \pi t)/\pi, & 0 \leq t < 1, \\ \int_{t-1}^t \sin \pi \tau d\tau = -\frac{2}{\pi} \cos \pi t, & t \geq 1. \end{cases}$$

Both pieces give $2/\pi$ at $t = 1$. After the pulse is fully inside, the output is a sinusoid at the input's frequency with amplitude $2/\pi$.

Zero-input and zero-state response

A system $Q(D)y = P(D)x$ with $Q(D) = D^n + a_{n-1}D^{n-1} + \dots + a_0$ needs n auxiliary conditions to pin down y . They are stated at $0-$, just before the input starts (Section 2.3.1): an input switched on at 0 can put a δ on the right-hand side, so $y(0)$ itself may be undefined. With $x(t) = 0$ for $t < 0$ and conditions $y^{(i)}(0-) = \alpha_i$, split the response in two (Section 2.3.2):

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$$\begin{aligned} y &= y_{zi} + y_{zs}, & y_{zi} : Q(D)y &= 0, & y^{(i)}(0-) &= \alpha_i, \\ & & y_{zs} : Q(D)y &= P(D)x, & y^{(i)}(0-) &= 0. \end{aligned}$$

The zero-input part is what the stored energy does with no input; the zero-state part is the input acting from rest, which is the convolution $y_{zs}(t) = \int_0^t x(\tau) h(t-\tau) d\tau$ of eq. (2.81). Convolution alone is the whole answer only when every α_i is 0, which is also the only case in which the system is linear.

The zero-input response

Replace D by a number λ and factor the characteristic polynomial $Q(\lambda)$ into its roots λ_i , each with a multiplicity m_i , the m_i adding up to n (Theorem 2.3.6). Each root contributes one term per unit of multiplicity (eqs. 2.66 and 2.72):

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$$\text{distinct: } y_{zi}(t) = \sum_{i=1}^n c_i e^{\lambda_i t}, \quad \text{repeated: } y_{zi}(t) = \sum_{i=1}^r \sum_{j=1}^{m_i} c_{i,j} t^{j-1} e^{\lambda_i t}.$$

Then differentiate $n-1$ times, set $t = 0-$, and solve the n equations $y_{zi}^{(i)}(0-) = \alpha_i$ for the constants (for distinct roots the matrix is Vandermonde, eq. 2.71). $P(D)$ plays no part: the input is zero. A conjugate pair $\sigma \pm j\omega$ with real conditions gives conjugate constants, and the two terms combine to $e^{\sigma t}(A \cos \omega t + B \sin \omega t)$ with A and B real.

When the conditions are given on a circuit's element variables rather than on y , turn them into $y(0-), y^{(1)}(0-), \dots$ first with the element laws $i = C dv/dt$ and $v = L di/dt$.

TUTORIAL 4 (SOLUTION), Q7 · MEDIUM · OFFICIAL

A source $x(t)$ drives a series loop of 1 H, 3Ω and $\frac{1}{2}$ F; the output is the loop current y . Find the zero-input response when the capacitor voltage has $v_c^{(1)}(0-) = 10$ V/s and $y^{(1)}(0-) = 20$ A/s.

DERIVATION · SKIP ON RE-READ

The series RLC equation with $L = 1$, $R = 3$, $1/C = 2$ gives $(D^2 + 3D + 2)y = Dx$, so $Q(\lambda) = (\lambda + 1)(\lambda + 2)$ and $y_{zi} = c_1 e^{-t} + c_2 e^{-2t}$. The capacitor carries the loop current, so $y(0-) = C v_c^{(1)}(0-) = \frac{1}{2} \cdot 10 = 5$. Then

$$c_1 + c_2 = 5, \quad -c_1 - 2c_2 = 20 \implies c_2 = -25, \quad c_1 = 30,$$

and $y_{zi}(t) = 30e^{-t} - 25e^{-2t}$ for $t \geq 0$. Check: $30 - 25 = 5$ and $-30 + 50 = 20$.

TUTORIAL 4 (SOLUTION), Q8 · MEDIUM · OFFICIAL

Find the zero-input response of $(D^2 + 4D + 4)y = (D + 1)x$ with $y(0-) = 1$, $y^{(1)}(0-) = -1$.

DERIVATION · SKIP ON RE-READ

$Q(\lambda) = (\lambda + 2)^2$: one root, -2 , of multiplicity 2, so

$$y_{zi} = (c_1 + c_2 t) e^{-2t}, \quad y_{zi}^{(1)} = (c_2 - 2c_1 - 2c_2 t) e^{-2t}.$$

At $0-$: $c_1 = 1$ and $c_2 - 2c_1 = -1$, so $c_2 = 1$ and

$$y_{zi}(t) = (1 + t) e^{-2t}, \quad t \geq 0.$$

The $(D + 1)$ on the input side was never used.

TUTORIAL 4 (SOLUTION), Q9 · MEDIUM · OFFICIAL

A source $x(t)$ drives a series loop of 1 H and 1 F; the output is the capacitor voltage y and the loop current is i . Find the zero-input response for $y(0-) = 1$, $i(0-) = 2$.

DERIVATION · SKIP ON RE-READ

KVL: $x = L i' + y$ with $i = C y' = y'$, so $(D^2 + 1)y = x$ and $\lambda = \pm j$. The current condition gives $y^{(1)}(0-) = i(0-)/C = 2$. From $y_{zi} = c_1 e^{jt} + c_2 e^{-jt}$: $c_1 + c_2 = 1$ and $j(c_1 - c_2) = 2$, so

$$y_{zi}(t) = \frac{1 - 2j}{2} e^{jt} + \frac{1 + 2j}{2} e^{-jt} = \cos t + 2 \sin t, \quad t \geq 0.$$

The constants are conjugates, as they must be for real conditions. The roots sit on the imaginary axis, so the stored energy swaps between L and C for ever: with no resistor, nothing dissipates it.

Stability

The same roots decide stability (Section 2.3.5). The system is *asymptotically stable* (Def 2.3.7) when $y_{zi} \rightarrow 0$ as $t \rightarrow \infty$ for every choice of $\alpha_0, \dots, \alpha_{n-1}$, and *BIBO stable* (Def 2.3.11) when every bounded input gives a bounded zero-state response.

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asymptotically stable $\iff \operatorname{Re}(\lambda_i) < 0$ for every root (Theorem 2.3.8),

BIBO stable $\iff m \leq n$ and $\operatorname{Re}(\lambda_i) < 0$ for every root (Theorem 2.3.12),

where m is the order of $P(D)$. Repeated roots do not change the verdict (Remark 2.3.9). Tutorial 4's Q7 $(-1, -2)$ and Q8 (-2 twice) are asymptotically stable; Q9 $(\pm j)$ is not, which is why its response never dies away.

Practice, with answers

1. (Worked for this lesson, easy, ai, from the Chapter 2 notes) Evaluate $\int e^{-\tau} \cos \tau \delta(t - \tau) d\tau$ and $\int \sin \tau \delta^{(2)}(t - \tau) d\tau$.

Answer: $e^{-t} \cos t$ and $-\sin t$.

2. (Worked for this lesson, medium, ai, from Tutorial 2) Convolve $x(t) = 2 - t$ on $[0, 2]$ with the unit pulse, 1 on $[0, 1]$.

Answer: $2t - t^2/2$ on $[0, 1]$, $5/2 - t$ on $[1, 2]$, $(3 - t)^2/2$ on $[2, 3]$, 0 elsewhere.

3. (Example 2.2.10, medium, ai-worked from the Chapter 2 notes) $x = t$ on $[0, 3]$, $y = 1$ on $[-1, 1]$, both 0 elsewhere. Find $x * y$.

Answer: $\alpha = \max(0, t - 1)$, $\beta = \min(3, t + 1)$, giving $(t + 1)^2/2$ on $[-1, 1]$, $2t$ on $[1, 2]$, $(9 - (t - 1)^2)/2$ on $[2, 4]$, and 0 elsewhere.

4. (Worked for this lesson, medium, ai, from Tutorial 4) Find the zero-input response of $(D^2 + 2D + 5)y = x$ with $y(0-) = 1$, $y^{(1)}(0-) = 1$, and say whether the system is asymptotically stable.

Answer: $\lambda = -1 \pm 2j$, so $y_{zi} = e^{-t}(A \cos 2t + B \sin 2t)$ with $A = 1$ and $-A + 2B = 1$: $y_{zi}(t) = e^{-t}(\cos 2t + \sin 2t)$. Both roots have real part -1 , so it is asymptotically stable.

Frequency-domain analysis

Trading calculus for algebra, and reading behaviour off poles.

The Laplace transform sends $x(t)$ to $X(s)$. Convolution becomes multiplication,

$$y = x * h \iff Y(s) = H(s) X(s),$$

with $H(s)$ (transform of h) the **transfer function**; and differentiation becomes multiplication by s , turning a differential equation into algebra. Getting back to time is partial-fraction expansion (Heaviside's method).

Writing $H(s)$ as a ratio of polynomials, the denominator's roots are the **poles**: the real part of a pole sets decay/growth, the imaginary part sets oscillation. A system is BIBO stable exactly when every pole has strictly negative real part.

Check yourself

1. A system has poles at $s = -1 \pm 3j$ and no others. Is it stable, and what does its response look like?

Answer. Stable: both poles have real part $-1 < 0$, strictly left half-plane. The response is an oscillation at 3 rad/s inside an e^{-t} envelope — it rings, and the ringing dies away.

Discrete time

The same two chapters again, with sums for integrals — and one thing that changes.

Linearity, time invariance, causality and BIBO stability transfer to sequences $x[n]$ unchanged. Sifting for sequences gives the **convolution sum**,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k],$$

and the z -transform repeats the previous section's move: $Y(z) = H(z)X(z)$, inverted back to a sequence by partial fractions.

TABLE 2 *continuous time versus discrete time*

CONTINUOUS TIME	DISCRETE TIME	CARRIES OVER?
Differential equation	Difference equation	Yes
Convolution integral	Convolution sum	Yes
Laplace transform, $H(s)$	z -transform, $H(z)$	Yes
Stable when all poles satisfy $\text{Re}(s) < 0$	Stable when all poles satisfy $ z < 1$	No

A pole at s contributes e^{st} (decays when $\text{Re}(s) < 0$); a pole at z contributes z^n (decays when $|z| < 1$) — the stable region is a half-plane in one setting and a disc in the other. A common error is carrying "poles in the left half-plane" across to the z -plane.

Check yourself

1. One system has poles at $z = 0.5$ and $z = 1.5$. Another has poles at $s = 0.5$ and $s = 1.5$. Which poles are the stable ones?

Answer. Only $z = 0.5$ is stable (magnitude < 1). Both s poles are unstable (real part > 0). Note $z = 0.5$ and $s = 0.5$ are the same number and land on opposite sides of the verdict — the plane you are in decides.

Periodic inputs

Handling a periodic input one harmonic at a time.

Feed an LTI system $e^{j\omega t}$ and the output is the same exponential scaled by a complex number depending on ω alone:

$$e^{j\omega t} \mapsto H(j\omega) e^{j\omega t}$$

($H(j\omega)$, the transfer function on the imaginary axis, is the frequency response). A periodic signal of period T_0 decomposes into a Fourier series at $\omega_0 = 2\pi/T_0$ and its whole multiples,

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}, \quad D_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt,$$

so the output is one complex multiplication per harmonic,

$$y(t) = \sum_{n=-\infty}^{\infty} D_n H(jn\omega_0) e^{jn\omega_0 t}.$$

Check yourself

1. A periodic signal's period is doubled. What happens to the spacing of the lines in its spectrum?

Answer. The spacing halves: the lines sit at multiples of $\omega_0 = 2\pi/T_0$, so doubling T_0 halves ω_0 . Letting that process continue – the period growing without bound – is the next section.

Non-periodic inputs

Letting the period run to infinity: filters, and radio.

As the period T_0 grows, the line spacing ω_0 shrinks to zero and the Fourier series becomes the Fourier transform:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt, \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega.$$

System analysis crosses the limit intact, $Y(\omega) = H(\omega) X(\omega)$. Designing $|H(\omega)|$'s shape is filtering; the ideal low-pass filter's impulse response is a sinc, nonzero for negative time, so it cannot be built. Amplitude modulation multiplies by $\cos(\omega_c t)$, which shifts the spectrum up to ω_c ; demodulation and a low-pass filter bring it back.

Check yourself

1. Why can the ideal low-pass filter not be built?

Answer. Its impulse response is a sinc, nonzero for negative time: the filter would have to respond before its input arrived, so it is not causal, and no physical system is non-causal.

Lesson sources: `src/ece207_ch1_signals_systems.jsx` (read 2026-10-05), `src/ece207_ch2_impulse_convolution.jsx` (read 2026-10-05) and `src/ece207_course_overview.jsx` (read 2026-09-27), and the companions built from the first and last, `lessons/ECE207/claude_lessons/ece207-ch1-signals-systems/ece207_ece207-ch1-signals-systems_companion.tex` and `lessons/ECE207/claude_lessons/ece207-course-overview/ece207_ece207-course-overview_companion.tex`; Prof. Andrew Heunis's lecture notes (Chapters 1–6), the class notes of September 17–26, and Tutorials 1–4 with their posted solutions (Tutorial 4's three answers checked against its solution and by substitution), filed under `lessons/ECE207/materials/`.