

Course notes: vector calculus to residues

Technical content only, organised by topic and deduplicated across the course's lessons – key formulas, worked examples, and short practice with answers. No lecture times, policies, grading or deadlines: those are in <https://lessons.ihsan.cc/ece206/ece206-course-overview/>.

1 Curves and surfaces

Weeks 1-2. Source: ece206-course-overview, weeks 1-2.

A curve is a map from an interval, $\vec{g}(t) = (x(t), y(t), z(t))$; a surface is a map from a plane region, $\vec{g}(u, v)$. Match a sum of two squares in an implicit equation to $\sin^2 + \cos^2 = 1$ to read off a parameterisation; a linear constraint is simply solved for the remaining coordinate.

ON THE FORMULA SHEET

$$\vec{v}(t) = \vec{g}'(t), \quad \text{speed} = \|\vec{g}'(t)\|, \quad s = \int_a^b \|\vec{g}'(t)\| dt.$$

$\vec{v}(t)$ is the velocity (tangent to the curve); its magnitude is the speed; arc length s integrates speed over the parameter interval.

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$$\vec{n} = \vec{g}_u \times \vec{g}_v, \quad S = \iint_D \|\vec{g}_u \times \vec{g}_v\| du dv.$$

$\vec{n}$ is normal to the surface (the order of the cross product fixes orientation); S integrates the area of the parallelogram spanned by the two tangents.

WORKED EXAMPLE --**ARC LENGTH (PROBLEM SET 1, Q5)**

Length of $\vec{g}(t) = t^3 \hat{i} + t^2 \hat{j} + t^3 \hat{k}$, $0 \leq t \leq 1$: $\vec{g}'(t) = 3t^2 \hat{i} + 2t \hat{j} + 3t^2 \hat{k}$, magnitude $\sqrt{18t^4 + 4t^2} = t\sqrt{18t^2 + 4}$ (legitimate since $t \geq 0$), so $s = \int_0^1 t\sqrt{18t^2 + 4} dt$. Substituting $u = 18t^2 + 4$, $du = 36t dt$ (limits 4 to 22):

$$s = \frac{1}{36} \int_4^{22} \sqrt{u} du = \frac{1}{54} (22^{3/2} - 8).$$

PRACTICE · ANSWERS ONLY · OFFICIAL

PS1 Q1: $\vec{r} = (2, 3, -1)$, $\vec{s} = (1, -2, 3)$: angle $\theta = 2\pi/3$, parallelogram area $7\sqrt{3}$. Lecture 3 p.12: surface area of the cone $\vec{g}(u, v) = (u \cos v, u \sin v, u)$, $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$: $S = \sqrt{2}\pi$. PS2 Q4: sphere $x^2 + y^2 + z^2 = 9$ between $z = 1$ and $z = 2$ has annulus $\sqrt{5} \leq r \leq \sqrt{8}$ in D_{xy} and $S = 6\pi$ (matches Archimedes' zone formula $2\pi Rh$).

2 Double integrals: definition, regions and order

Week 3. Source: multiple-integrals, Lecture 4 pp.1-7.

$$\iint_D f(x, y) dA = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i, y_i) \Delta A_i.$$

With $f = 1$ this gives the area of D ; with f a surface density it gives the total mass or charge. On a rectangle the order doesn't matter, and a product integrand splits:

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$$\iint_D f dA = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f dx dy = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f dy dx, \quad \iint_D g(x)h(y) dA = \int_{y_1}^{y_2} h dy \cdot \int_{x_1}^{x_2} g dx.$$

Two descriptions of a plane region fix the order of integration – the variable with function limits is integrated first:

$$(1) x_\ell \leq x \leq x_u, \quad y_\ell(x) \leq y \leq y_u(x) \qquad (2) y_\ell \leq y \leq y_u, \quad x_\ell(y) \leq x \leq x_u(y).$$

To switch order, redraw the region from the limits you have and describe it the other way – never swap limits in place.

**WORKED EXAMPLE --
FORCING A SWITCH
(LECTURE 4 P.7)**

$\int_0^1 \int_x^1 e^{-y^2} dy dx$: e^{-y^2} has no elementary antiderivative, so switch. The triangle $0 \leq x \leq 1$, $x \leq y \leq 1$ is also $0 \leq y \leq 1$, $0 \leq x \leq y$.

$$\int_0^1 \int_0^y e^{-y^2} dx dy = \int_0^1 y e^{-y^2} dy = \left[-\frac{1}{2} e^{-y^2} \right]_0^1 = \frac{1 - e^{-1}}{2} \approx 0.316.$$

PRACTICE -- PROBLEM SET 2, Q1 · CORE · OFFICIAL

- (a) $\iint_D (x + y) dA$, triangle $(0, 0), (0, 1), (2, 1)$: $\frac{4}{3}$. (b) $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{1 + x^3} dx dy$: $\frac{2}{9}(2\sqrt{2} - 1)$.
 (c) $\iint_D (x^2 + y^2) dA$, D bounded by $y = x^2$, $x = y^2$: $\frac{6}{35}$. (d) $\int_{-2}^0 \int_{-2}^x \frac{x}{\sqrt{x^2 + y^2}} dy dx$: $2(1 - \sqrt{2})$.

3 Change of variables and polar coordinates

Week 3. Source: multiple-integrals, Lecture 4 pp.8-13.

The Jacobian is the local area scale factor of a map $(u, v) = F(x, y)$; the Jacobian of the inverse map is its reciprocal:

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$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}, \quad \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u, v)}{\partial(x, y)}}.$$

Theorem. For $(x, y) = F(u, v) = (f(u, v), g(u, v))$ mapping D_{uv} one-to-one onto D_{xy} , with G continuous on D_{xy} :

ON THE FORMULA SHEET

$$\iint_{D_{xy}} G(x, y) dx dy = \iint_{D_{uv}} G(f(u, v), g(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv.$$

Pick u, v so each piece of the boundary is a constant, which makes D_{uv} a rectangle.

WORKED EXAMPLE --
THE FOUR-CURVES
REGION (LECTURE 4
P.11)

D bounded by $y = 1/x$, $y = 2/x$, $y = x^2$, $y = 2x^2$: let $u = xy$, $v = y/x^2$, giving $1 \leq u \leq 2$, $1 \leq v \leq 2$.

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{3y}{x^2} = 3v \implies \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \frac{1}{3v}.$$

$$A(D_{xy}) = \frac{1}{3} \int_1^2 \int_1^2 \frac{1}{v} du dv = \frac{1}{3} \ln 2.$$

PRACTICE -- PROBLEM SET 2, Q5 · CORE · OFFICIAL

$(u, v) = F(x, y) = (y/x^2, x/y^2)$, $x, y > 0$. $\partial(x, y)/\partial(u, v) = \frac{1}{3u^2v^2}$; area of $D_{xy} = \{x^2 \leq y \leq 2x^2, 2y^2 \leq x \leq 3y^2, x, y > 0\}$ is $\frac{1}{36}$.

Polar is the Theorem with $(x, y) = (r \cos \theta, r \sin \theta)$, Jacobian r :

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$$\frac{\partial(x, y)}{\partial(r, \theta)} = r, \quad \iint_{D_{xy}} G(x, y) dx dy = \iint_{D_{r\theta}} G(r \cos \theta, r \sin \theta) r dr d\theta.$$

Use polar when the boundary is circles about the origin and rays from it, or the integrand is built from $x^2 + y^2$.

WORKED EXAMPLE --
LECTURE 4 P.13

$D = \{1 \leq x^2 + y^2 \leq 4, x \geq 0, y \geq 0, y \leq x\}$ is $1 \leq r \leq 2, 0 \leq \theta \leq \pi/4$.

$$\begin{aligned} \iint_D xy dA &= \int_0^{\pi/4} \int_1^2 r^3 \cos \theta \sin \theta dr d\theta \\ &= \left[\frac{r^4}{4} \right]_1^2 \cdot \left[\frac{\sin^2 \theta}{2} \right]_0^{\pi/4} = \frac{15}{4} \cdot \frac{1}{4} = \frac{15}{16}. \end{aligned}$$

PRACTICE -- PROBLEM SET 2, Q2, Q3, Q6 · INTRO core / challenge · official

Q2: volume under $z = 2 - x^2 - y^2$, above $z = 0$, inside $x^2 + y^2 = 1$: $\frac{3\pi}{2}$. Q3:

$\iint_D (x^2 + y^2)^{3/2} dA$, $D = \{1 \leq x^2 + y^2 \leq 4, y \geq \sqrt{3}x, y \geq x/\sqrt{3}\}$: $\frac{31\pi}{6}$. Q6: $\iint_D x^2 dA$, D the ellipse $10x^2 + 6xy + y^2 = 2$ (hint: $u = x/\sqrt{2}$, $v = (3x + y)/\sqrt{2}$ maps it to the unit disk): π .

4 Triple integrals; cylindrical and spherical coordinates

Week 3. Source: multiple-integrals, Lecture 4 pp.14-16, Lecture 5, Lecture 6 pp.1-10, primer 2.

$$\iiint_D f(x, y, z) dV = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i, y_i, z_i) \Delta V_i.$$

With D between $z_\ell(x, y)$ and $z_u(x, y)$ over the shadow D_{xy} :

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$$\iiint_D f(x, y, z) dV = \iint_{D_{xy}} \int_{z_\ell(x, y)}^{z_u(x, y)} f(x, y, z) dz dA.$$

Any of x, y, z may be innermost, giving $3! = 6$ possible orders; each inner limit may depend only on the variables outside it.

WORKED EXAMPLE --
THE TETRAHEDRON
(LECTURE 5 P.4)

$\iiint_D z dV$ over the tetrahedron with vertices
 $(a, 0, 0), (0, b, 0), (0, 0, c), (0, 0, 0)$, bounded by the coordinate planes and
 $x/a + y/b + z/c = 1$:

$$0 \leq z \leq c\left(1 - \frac{x}{a} - \frac{y}{b}\right), \quad 0 \leq y \leq b\left(1 - \frac{x}{a}\right), \quad 0 \leq x \leq a.$$

Substituting $x = as$, $y = bw$ turns the shadow into the unit triangle
 $s + w \leq 1$ with $dx dy = ab ds dw$:

$$\iiint_D z dV = \frac{c^2}{2} ab \int_0^1 \int_0^{1-s} (1 - s - w)^2 dw ds = \frac{abc^2}{2} \cdot \frac{1}{12} = \frac{abc^2}{24}.$$

PRACTICE -- PROBLEM SET 2, Q7 · CORE · OFFICIAL

Volume of the tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$, $x + 2y + 2z - 4 = 0$: $\frac{8}{3}$.

Cylindrical keeps z and replaces x, y by polar coordinates, Jacobian r ; spherical names a point by $\rho \geq 0$ (distance from origin), ϕ (angle from $+z$ -axis) and θ (as in cylindrical), Jacobian $\rho^2 \sin \phi$:

ON THE FORMULA SHEET

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z, \quad dV = r dr d\theta dz.$$

ON THE FORMULA SHEET

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi, \quad dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

Use cylindrical for cylinders/paraboloids about the z -axis; spherical for spheres/cones about the z -axis. A plane becomes a variable bound in spherical coordinates: $z = c$ is $\rho = c / \cos \phi$.

WORKED EXAMPLE --
UNIT BALL ABOVE A
CONE (PRIMER 2 P.28,
LECTURE 6 PP.4-10)

Volume between the cone $\phi = \pi/4$ and the xy -plane ($\phi = \pi/2$), inside the unit sphere:

$$V = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 2\pi \cdot \frac{\sqrt{2}}{2} \cdot \frac{1}{3} = \frac{\sqrt{2}\pi}{3}.$$

Lecture 6 pp.9-10 checks the same solid in cylindrical, in both orders, and recovers the same $\sqrt{2}\pi/3$.

PRACTICE -- PROBLEM SET 2, Q8, Q9, Q10 · CORE *core / challenge · official*

Q8: mass of S inside $x^2 + y^2 = 1$, below $z = 4$, above $z = 1 - x^2 - y^2$, density $K\sqrt{x^2 + y^2}$: $\frac{12\pi K}{5}$. Q9: mass of the region between the upper half of $x^2 + y^2 + z^2 = 4$ and the cone $z = \frac{1}{\sqrt{3}}\sqrt{x^2 + y^2}$, density $2 - z$: $\frac{7\pi}{3}$. Q10: volume of the sphere $x^2 + y^2 + z^2 = 4$ cut by $z = 1$ (the part $z \geq 1$): $\frac{5\pi}{3}$.

5 Vector fields and field lines

Week 4. Source: line-integrals, Lecture 6 pp.11-14, Lecture 7 pp.1-10, primer 3 pp.1-7.

A scalar field attaches a number to each point, $f : \mathbb{R}^n \rightarrow \mathbb{R}$; a vector field attaches a vector, $\vec{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (Lecture 6 p.13). Every scalar field has an associated vector field, its gradient, pointing in the direction f increases fastest with that rate of increase as its length:

ON THE FORMULA SHEET

$$f(x, y) = x^2 y^3 \implies \nabla f = (2xy^3, 3x^2 y^2).$$

A temperature gradient drives heat flow, a pressure gradient drives fluid flow, a concentration gradient drives diffusion, each flowing down its gradient; the electric field is the same relation with a sign, $\vec{E} = -\nabla V$. Reading an arrow plot: evaluate a candidate formula at a couple of

points and compare directions – $(y, 0)$ gives horizontal arrows; (x, y) is a point source and $(-x, -y)$ a sink (Lecture 7 p.9); $(x, -y)$ a saddle (flow in along one axis, out along the other).

The **field lines** of $\vec{F}$ are the curves to which $\vec{F}$ is tangent at every point – streamlines, in a fluid. Writing the curve as $\vec{g}(t)$:

$$\vec{g}'(t) = \vec{F}(\vec{g}(t)).$$

In the plane, with $\vec{F} = (F_1, F_2)$, this splits into $dx/dt = F_1$, $dy/dt = F_2$; the chain rule then removes t (Lecture 7 pp.5-6):

ON THE FORMULA SHEET

$$\frac{dy}{dx} = \frac{F_2(x, y)}{F_1(x, y)}.$$

The parameter drops out: the *shape* of a field line depends only on the direction of $\vec{F}$, not its length; the *direction* of travel must be read back off the field at one point.

WORKED EXAMPLE --

FIELD LINES OF

$F = (y, -x)$ (PRIMER P.4,
LECTURE 7 P.7)

$F_1 = y$, $F_2 = -x$, so $dy/dx = -x/y \Rightarrow y dy = -x dx \Rightarrow x^2 + y^2 = C$: circles about the origin. $F_1 = y$ moves right above the x -axis, $F_2 = -x$ moves down right of the y -axis, so the circles are traversed clockwise. $(-y, x)$ gives the same circles counterclockwise (PS3 Q7a). Lecture 7 p.8: a float on $(y, 0)$ rotates clockwise.

PRACTICE -- PROBLEM SET 3, Q1, Q2, Q12(B) • INTRO core / core • official

Q1: $\vec{F} = (x, x^2)$: $dy/dx = x$, $y = x^2/2 + C$, parabolas running upward; the y -axis is a line source. Q2: $\vec{E} = kq \vec{r}/r^3$: $|y| = c|x|$ rotated, rays out of the origin (a source; a sink if $q < 0$). Q12(b): $\phi = \frac{1}{2} \ln(x^2 + y^2)$: equipotentials circles $x^2 + y^2 = e^A$, field lines $y = Bx$, orthogonal.

6 Line integrals: scalar and work

Week 4. Source: line-integrals, Lecture 7 pp.11-15, primer 3 pp.8-14 (C_1 and C_2 below are the lesson's working).

A wire has the shape of a curve $\vec{x} = \vec{g}(t)$ and density $f(\vec{x})$; refining a Riemann sum over its arc length gives the scalar line integral:

ON THE FORMULA SHEET

$$\int_C f \, ds = \int_a^b f(\vec{g}(t)) \|\vec{g}'(t)\| \, dt, \quad C: \vec{x} = \vec{g}(t), \, a \leq t \leq b.$$

With $f = 1$ this is the arc length of C . The speed $\|\vec{g}'\|$ makes the answer independent of parameterisation and of direction: a scalar line integral is a property of the curve as a set of points.

WORKED EXAMPLE --

Wire along $x^2 + y^2 = a^2$ in the first quadrant, density $f = x + y$.

THE QUARTER-CIRCLE

Parameterise $\vec{g}(t) = (a \cos t, a \sin t)$, $0 \leq t \leq \pi/2$, speed a :

WIRE (PRIMER P.9,

LECTURE 7 P.12)

$$M = \int_0^{\pi/2} (a \cos t + a \sin t) a \, dt = a^2 [\sin t - \cos t]_0^{\pi/2} = a^2(1 - (-1)) = 2a^2.$$

PRACTICE -- PROBLEM SET 3, Q3, Q4, Q5 · INTRO *core/core · official*

Q3: $f = 2xy$ on $(t, t+1)$, $0 \leq t \leq 4$: $\frac{176\sqrt{2}}{3}$. Q4: helix $(R \cos t, R \sin t, \frac{h}{2\pi}t)$, density $\rho_0(1 + (k-1)z/h)$: $M = \frac{(k+1)\rho_0}{2} \sqrt{4\pi^2 R^2 + h^2}$. Q5: charge z on $(t \cos t, t \sin t, t)$, $0 \leq t \leq 4$: $\|\vec{g}'\| = \sqrt{2+t^2}$, $Q = \frac{52\sqrt{2}}{3}$.

For a vector field, integrating force along a directed path gives work. Lecture 7 pp.13-15 builds it from $W = (\vec{F} \cdot \hat{i})(b-a)$ for a constant force on a line, through the work element $\Delta W \approx (\vec{F}(x) \cdot \hat{i})\Delta x$, to $\Delta W \approx \vec{F}(\vec{g}(t)) \cdot \vec{g}'(t)\Delta t$:

ON THE FORMULA SHEET

$$\int_C \vec{F} \cdot d\vec{x} = \int_a^b \vec{F}(\vec{g}(t)) \cdot \vec{g}'(t) \, dt = \int_C P \, dx + Q \, dy + R \, dz, \quad \vec{F} = (P, Q, R).$$

Unlike the scalar integral, the work integral carries the *velocity* $\vec{g}'$, not just its length: reversing the direction of travel reverses the sign.

WORKED EXAMPLE --

THREE PATHS, ONE

FIELD (PRIMER P.13-14:

 $F=(Y, KX)$

From $(0, 0)$ to $(1, 1)$: C_1 the parabola $y = x^2$, C_2 the line $y = x$, C_3 up the y -axis then along $y = 1$. Primer p.14 gives $C_3 = 0 + 1 = 1$ for every k (perpendicular leg contributes nothing); C_1 and C_2 are the lesson's working:

$$C_1 : \int_0^1 (t^2, kt) \cdot (1, 2t) dt = \int_0^1 (1 + 2k)t^2 dt = \frac{1 + 2k}{3}.$$

$$C_2 : \int_0^1 (t, kt) \cdot (1, 1) dt = \int_0^1 (1 + k)t dt = \frac{1 + k}{2}.$$

In general the three differ; all three equal 1 only at $k = 1$, where $\vec{F} = (y, x)$ – a "special type of field" the next section names.

PRACTICE -- PROBLEM SET 3, Q6, Q7(B), Q8 · INTRO *core / core · official*

Q6: (x, y, z) along $y = x^2$, x from -1 to 2 : $W = 9$. Q7(b): $(-y, x)$ counterclockwise round $x^2/a^2 + y^2/b^2 = 1$: $2\pi ab$ (a closed loop with a nonzero answer, so no potential). Q8: (x^3, y, z) round the circle $(0, a \cos t, a \sin t)$: $W = 0$, the field's rays meet the circle at right angles.

7 Conservative fields, potentials and the FTLI

Weeks 4-5. Source: line-integrals, primer 3 pp.15-23, Lecture 8 pp.5-16.

$\vec{F} = (y, x) = \nabla(xy)$: a field with a scalar ϕ whose gradient it is is a **gradient** or **conservative** field, ϕ its **potential**, and its line integrals are **path independent** (Lecture 8 p.7). (Physics uses the opposite sign: $\vec{E} = -\nabla V$.) The Fundamental Theorem for Line Integrals (Lecture 8 p.8): if $\vec{F} = \nabla\phi$ and C runs from $\vec{a}$ to $\vec{b}$,

ON THE FORMULA SHEET

$$\int_C \vec{F} \cdot d\vec{x} = \int_C \nabla\phi \cdot d\vec{x} = \phi(\vec{b}) - \phi(\vec{a}).$$

It reads like the FTC, $\int_a^b f'(x) dx = f(b) - f(a)$, with ϕ as an "antiderivative" of $\vec{F}$, and it needs no parameterisation of C .

DERIVATION · SKIP ON RE-READ

Lecture 8 p.9. Along $\vec{g}(t) = (x(t), y(t), z(t))$, $\nabla\phi \cdot \vec{g}'$ is exactly the chain-rule expansion of $d\phi/dt$:

$$\int_C \nabla\phi \cdot d\vec{x} = \int_a^b (\phi_x x' + \phi_y y' + \phi_z z') dt = \int_a^b \frac{d\phi}{dt} dt = \phi(\vec{b}) - \phi(\vec{a}).$$

If C is a simple closed curve (one that does not cross itself), start and end coincide, so (Lecture 8 p.10)

$$\oint_C \vec{F} \cdot d\vec{x} = 0 \quad (\vec{F} = \nabla\phi, C \text{ simple, closed}).$$

Testing for ϕ (Lecture 8 p.11): if $\vec{F} = (F_1, F_2) = (\phi_x, \phi_y)$ then the mixed partials of ϕ agree, so

ON THE FORMULA SHEET

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

is *necessary* (the lecture boxes it as “if $\vec{F} = \nabla\phi$, this is true” and states no converse). It is also sufficient on a region with no holes in it (simply connected) – the standard counterexample, $(-y, x)/(x^2 + y^2)$, passes the test everywhere but the origin and still circulates 2π around the unit circle (the lesson’s note, not the lecture’s; matters only for fields that blow up somewhere). A test that passes means only the endpoints matter: the Lecture 8 p.15 poll compares four curves by their endpoints alone, without computing ϕ or any integral.

WORKED EXAMPLE --

FINDING ϕ (LECTURE 8
PP.12-14)

$\vec{F} = (2xe^y + y, x^2e^y + x - 2y)$: test $\partial F_1/\partial y = 2xe^y + 1 = \partial F_2/\partial x$ (defined everywhere, so ϕ exists). Integrate F_1 in x with $h(y)$ free, match against F_2 :

$$\phi = \int (2xe^y + y) dx = x^2e^y + xy + h(y).$$

$$x^2e^y + x + h'(y) = x^2e^y + x - 2y \Rightarrow h = -y^2 + K,$$

$$\phi(x, y) = x^2e^y + xy - y^2 + K.$$

(This is the method for exact first-order ODEs from ECE205:

$M dx + N dy = 0$ is exact when $M_y = N_x$, solution $\phi = C$.) The theorem then integrates the same field along the unit quarter circle $(1, 0) \rightarrow (0, 1)$ counter-clockwise in one line; parameterising it instead gives an integrand the lecture abandons as messy:

$$\int_C \vec{F} \cdot d\vec{x} = \phi(0, 1) - \phi(1, 0) = (0 + 0 - 1) - (1 + 0 - 0) = -2.$$

The level curves $\phi(x, y) = \text{const}$ are the **equipotential** curves (Lecture 8 p.16); since the gradient at a point is perpendicular to the level curve through it, field lines of $\vec{F} = \nabla\phi$ cross every equipotential at a right angle (for $\phi = xy$: equipotentials $xy = c$, field lines $x^2 - y^2 = \text{const}$). For an electric field this says $\vec{E}$ is perpendicular to the curves of constant voltage. In $\mathbb{R}^3$ the level sets $\phi(x, y, z) = \text{const}$ are **equipotential surfaces**, and field lines cross

them at right angles too. For a point or line charge, field lines are rays out from the charge and equipotentials are circles (spheres/cylinders in space); Lecture 8 p.16 draws the dipole.

PRACTICE -- PROBLEM SET 3, Q9-Q12(A), Q13 · CORE *core / stretch / core / stretch · official*

Q9: $(y, x) = \nabla(xy)$ from $(0, 0)$ to $(1, 1)$: 1. Q10: $(ye^{xy}, xe^{xy} - 2y)$, $\phi = e^{xy} - y^2$, from $(0, 1)$ to $(1, 0)$: 1. Q11: $\phi = y^2 \sin x + xz^3 - 4y + z^2/2$ (all three mixed-partial pairs agree), from $(0, 1, -1)$ to $(\frac{\pi}{2}, -1, 2)$: $4\pi + \frac{21}{2}$. Q12(a): $(x, y)/(x^2 + y^2) = \nabla \frac{1}{2} \ln(x^2 + y^2)$. Q13: with $P = -\phi$, $\frac{d}{dt}P(\vec{g}(t)) = -\|\nabla P\|^2 < 0$ along a field line, and $P + \frac{1}{2}mv^2$ is constant.

8 Green's theorem

Week 5. Source: Lecture 8 pp.17-18, Lecture 9 pp.1-8, primer 4 pp.1-8.

Theorem (Lecture 8 p.17). Let D be a region in the plane whose boundary ∂D is a smooth, simple curve oriented counterclockwise. If $\vec{F} = (F_1, F_2)$ is C^1 on D and ∂D (both components have continuous first partials), then

ON THE FORMULA SHEET

$$\oint_{\partial D} \vec{F} \cdot d\vec{x} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA.$$

Like the FTLI it is an “integral of a derivative” theorem, a generalisation of the FTC: some integration on one side cancels against the derivatives of $\vec{F}$ on the other. Where $\partial F_2/\partial x - \partial F_1/\partial y = 0$ – the same test as for a potential – the circulation round the curve is zero, but only under the theorem’s hypotheses: $\vec{F}$ must be C^1 on all of D . (Lecture 9 p.1 ties a caution about Problem Set 4 Q3 to exactly this claim and leaves it unfinished on the page; the field $(-y, x)/(x^2 + y^2)$ of the last section is the case it guards against.)

Interpretation (Lecture 9 p.3, primer 4 p.3). When C is closed, $\oint_C \vec{F} \cdot d\vec{x}$ is the **circulation** of $\vec{F}$ around C . The integrand is circulation per unit area, the **vorticity**: the local rotation about an axis perpendicular to the xy -plane, positive for counterclockwise rotation. Green’s theorem says the global circulation around a curve is the sum of the local circulation elements inside it; neighbouring elements cancel along shared edges, so only the boundary survives. Lecture 9 p.4 (a paddle-wheel demo, with D a disk centred on the origin): $(y^2, 0)$ has vorticity $-2y$, which integrates to 0 over D by symmetry, so its circulation is zero although paddle wheels above the axis turn clockwise and those below counterclockwise; $(y, 0)$ has vorticity -1 everywhere, so its circulation is $-\text{area}(D)$, and every wheel turns clockwise, as the float did in Lecture 7 p.8.

POLL -- READING
VORTICITY (LECTURE 9
P.5)

$\vec{F} = (y^2, -y)$: vorticity $0 - 2y = -2y$, negative above the x -axis and positive below. So a small disk in the flow turns clockwise in quadrants 1 and 2 and counterclockwise in 3 and 4. The circulation round $(x-1)^2 + y^2 = 1$ is zero (the disk is symmetric about $y = 0$); round $x^2 + (y+1)^2 = 1$ it is not (the disk lies below the axis, where $-2y > 0$).

WORKED EXAMPLE --
THE TRIANGLE
(LECTURE 8 P.18,
LECTURE 9 P.2)

$\oint_C \vec{F} \cdot d\vec{x}$ for $\vec{F} = (xy, 2xy)$, C the triangle $(0,0) \rightarrow (1,0) \rightarrow (0,1) \rightarrow (0,0)$, counterclockwise. Without Green's theorem the three sides are parameterised separately. With it, over $D : 0 \leq x \leq 1, 0 \leq y \leq 1-x$:

$$\begin{aligned}\oint_C \vec{F} \cdot d\vec{x} &= \iint_D (2y - x) dA = \int_0^1 \int_0^{1-x} (2y - x) dy dx \\ &= \int_0^1 (1 - 3x + 2x^2) dx = \frac{1}{6}.\end{aligned}$$

Area (Lecture 9 p.6, primer 4 p.6). If $\partial F_2/\partial x - \partial F_1/\partial y = 1$ the right side is $\iint_D dA$, so a line integral measures area; $\vec{F} = (0, x)$ is one such field (the planimeter, Lecture 9 p.7, works on this idea: trace any closed curve and read off the area inside):

$$A(D) = \oint_{\partial D} (0, x) \cdot d\vec{x} = \oint_{\partial D} x dy.$$

For the ellipse $x^2/a^2 + y^2/b^2 = 1$, parameterise ∂D as $\vec{g}(t) = (a \cos t, b \sin t)$, $0 \leq t \leq 2\pi$ (Lecture 9 p.6, answer πab ; the details are left as an exercise there): $\int_0^{2\pi} a \cos t \cdot b \cos t dt = \pi ab$. Problem Set 3 Q7(b)'s $2\pi ab$ for $(-y, x)$ is the same fact: that field has vorticity 2.

Three ways to evaluate $\int_C \vec{F} \cdot d\vec{x}$ (Lecture 9 p.8, primer 4 p.8): parameterise C and integrate; if $\vec{F} = \nabla \phi$, take $\phi(\vec{b}) - \phi(\vec{a})$; if C is closed, use Green's theorem. Knowing which applies is the skill.

9 Surface integrals and flux

Week 5. Source: Lecture 9 pp.9-14, Lecture 10 pp.1-3, primer 4 pp.9-14.

The scalar surface integral is the surface analogue of $\int_C f ds$, with $\|\vec{g}_u \times \vec{g}_v\|$ in the place of $\|\vec{g}'(t)\|$:

ON THE FORMULA SHEET

$$\iint_{\Sigma} f \, dS = \iint_{D_{uv}} f(\vec{g}(u, v)) \|\vec{g}_u \times \vec{g}_v\| \, du \, dv, \quad \Sigma : \vec{x} = \vec{g}(u, v), \, (u, v) \in D_{uv}.$$

The key step is parameterising Σ . With $f = 1$ it is the surface area of the first section; if f is a mass or charge density, it is the total mass or charge (Lecture 9 p.9).

Flux measures how much of a physical quantity crosses a fixed surface per unit time (Lecture 9 pp.11-12). Mass carried by a fluid of velocity $\vec{v}(\vec{x})$ and density $\rho(\vec{x})$ has mass-flux vector $\vec{F} = \rho\vec{v}$; heat flows from hot to cold, driven by the temperature gradient, with heat-flux vector $\vec{F} = -K\nabla T$, K the thermal conductivity. Electric flux, magnetic flux and mass diffusing down a concentration gradient are the others. A flux element depends on $\vec{F}$ and on the surface's orientation, given by its unit normal $\hat{n}$ (Lecture 9 pp.10, 13):

$$\Delta(\text{flux}) \approx (\vec{F} \cdot \hat{n}) \Delta S, \quad \Delta S \approx \|\vec{g}_u \times \vec{g}_v\| \Delta u \Delta v, \quad \hat{n} = \frac{\vec{g}_u \times \vec{g}_v}{\|\vec{g}_u \times \vec{g}_v\|}.$$

Substituting the second and third into the first, the lengths cancel:

ON THE FORMULA SHEET

$$\iint_{\Sigma} \vec{F} \cdot \hat{n} \, dS = \iint_{D_{uv}} \vec{F}(\vec{g}(u, v)) \cdot (\vec{g}_u \times \vec{g}_v) \, du \, dv,$$

also written $\iint_{\Sigma} \vec{F} \cdot d\vec{S}$. Steps: parameterise Σ ; compute $\vec{g}_u$, $\vec{g}_v$ and $\vec{g}_u \times \vec{g}_v$; dot $\vec{F}(\vec{g}(u, v))$ with it; integrate over D_{uv} . The definition uses the *outward* normal, so check that $\vec{g}_u \times \vec{g}_v$ points outward and swap the order (or the sign) if it does not (Lecture 9 p.13).

ORIENTATION CHECK --

THE CYLINDER

(LECTURE 9 P.14,

PRIMER 4 P.14;

PROBLEM SET 4 Q5)

$x^2 + z^2 = b^2$, $-\ell \leq y \leq \ell$, parameterised $\vec{g}(u, v) = (b \cos u, v, b \sin u)$, $0 \leq u \leq 2\pi$, $-\ell \leq v \leq \ell$. Compare the normal with $\vec{w} = (-a, 0, -c)$, which points from a point $(a, 0, c)$ toward the axis:

$$\vec{g}_u = (-b \sin u, 0, b \cos u), \quad \vec{g}_v = (0, 1, 0),$$

$$\vec{g}_u \times \vec{g}_v = (-b \cos u, 0, -b \sin u),$$

a positive multiple of $\vec{w}$: this order gives the *inward* normal. For outward flux use $\vec{g}_v \times \vec{g}_u = -(\vec{g}_u \times \vec{g}_v)$, i.e. swap u and v or multiply by -1 .

10 Divergence and curl

Week 5. Source: Lecture 10 pp.4-11, Lecture 11 pp.1-5, primer 4 pp.15-25; identities from primer 5 pp.17-18.

Green's theorem has two natural extensions to $\mathbb{R}^3$: a closed curve becomes a closed surface $\partial\Omega$ around a solid Ω , or the plane region is stretched into an open surface Σ with the same boundary curve $\partial\Sigma$. Each needs a derivative of a vector field to put in the integrand. A field $\vec{F} = (F_1, F_2, F_3)$ on $\mathbb{R}^3$ has nine first partials, each F_i in x, y and z ; divergence and curl are two combinations of them. The operator **del** (nabla), $\nabla = (\partial_x, \partial_y, \partial_z)$, applied to a scalar f gives the **gradient** $\nabla f = (\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$, and applied to $\vec{F}$ it gives the two, through the dot and the cross product (Lecture 10 pp.4-6):

ON THE FORMULA SHEET

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z},$$

ON THE FORMULA SHEET

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right).$$

The divergence is a scalar field; $\nabla \cdot \vec{F} = 0$ means $\vec{F}$ is **incompressible (solenoidal)**. The curl is a vector field; $\nabla \times \vec{F} = \vec{0}$ means $\vec{F}$ is **irrotational**. (Primer 4 p.18 prints $\partial/\partial z$ first in the determinant's middle row; it should be $\partial/\partial x$, and its expanded vector is right.) For a planar field $(F_1(x, y), F_2(x, y), 0)$ the curl is $(0, 0, \partial F_2/\partial x - \partial F_1/\partial y)$: Green's vorticity, so curl measures local circulation. ∇ needs something to act on: $f\nabla$, $\vec{F} \cdot \nabla$ and $\vec{F} \times \nabla$ leave the partials hanging and are undefined here.

WORKED EXAMPLE --

$$\vec{F} = (xy, y^2 + z, x^2):$$

LECTURE 10 P.8

$$\nabla \cdot \vec{F} = y + 2y + 0 = 3y, \quad \nabla \times \vec{F} = (0 - 1, 0 - 2x, 0 - x) = (-1, -2x, -x).$$

Product rules (Lecture 10 pp.10-11). The naive guess $(\nabla \cdot f)\vec{G} + \dots$ fails, because a scalar has no divergence; the gradient takes its place:

ON THE FORMULA SHEET

$$\nabla \cdot (f\vec{G}) = \nabla f \cdot \vec{G} + f(\nabla \cdot \vec{G}), \quad \nabla \times (f\vec{G}) = \nabla f \times \vec{G} + f(\nabla \times \vec{G}).$$

The first comes from the ordinary product rule in each slot (Lecture 10 p.10): $\sum_i \partial_i (f G_i) = \sum_i (\partial_i f) G_i + f \sum_i \partial_i G_i$. Two identities, from Problem Set 4 Q10 as primer 5 cites them: $\nabla \times (\nabla \phi) = \vec{0}$ (curl of a gradient) and $\nabla \cdot (\nabla \times \vec{F}) = 0$ (divergence of a curl). Lecture 11 p.3 checks which expressions are defined, for a scalar f and a vector field $\vec{G}$: $\nabla \times (\nabla f)$ and $\nabla \times (\nabla \times \vec{G})$ are vector fields and $\nabla \cdot (\nabla \times \vec{G})$ is a scalar, while $\nabla \times f$, $\nabla \cdot (\vec{G} \times \nabla)$ and $\nabla \cdot (\nabla f \cdot \vec{G})$ are not defined ($\nabla f \cdot \vec{G}$ is a scalar).

Radial fields $\vec{F} = f(r) \vec{r}$, with $\vec{r} = (x, y, z)$ and $r = \|\vec{r}\| = \sqrt{x^2 + y^2 + z^2}$: such a field points straight toward or away from the origin, and its size depends only on the distance to it (Lecture 10 p.9). The field of a point charge, $\vec{r}/r^3 = (x, y, z)/(x^2 + y^2 + z^2)^{3/2}$, is the case $f = 1/r^3$. Lecture 11 pp.4-5 work out the facts, with a chain rule for ∇ ($\nabla f = \frac{df}{dr} \nabla r$, from $f \rightarrow r \rightarrow x, y, z$); $\nabla \times (f\vec{r}) = \vec{0}$ is left as an exercise:

$$\begin{aligned} \nabla \cdot \vec{r} &= 3, & \nabla \times \vec{r} &= \vec{0}, & \frac{\partial r}{\partial x} &= \frac{x}{r} \Rightarrow \nabla r = \frac{\vec{r}}{r}, & \nabla f(r) &= f'(r) \frac{\vec{r}}{r}. \\ \nabla \cdot (f\vec{r}) &= f'(r) \frac{\vec{r} \cdot \vec{r}}{r} + 3f = rf'(r) + 3f(r). \end{aligned}$$

The lecture's exercise, $\nabla \cdot (f\vec{r}) = 0$ (left blank; these notes' working), gives $rf' = -3f$, so $f = C/r^3$: the point-charge field is the radial field with no divergence away from the origin.

11 The Divergence Theorem

Week 7 in the outline. Source: Lecture 11 pp.6-17; primer 5 pp.1-11; Problem Set 5 Q1-3.

A closed curve (a loop) has no endpoints and bounds a plane region; an open curve's boundary is its endpoints. Likewise a closed surface (a sphere) has no edge and bounds a solid, and an open surface (a hemisphere) has a curve as its boundary. The Divergence Theorem lifts Green's theorem to closed surfaces, Stokes' theorem to open ones.

Theorem (Gauss). Let Ω be a region in $\mathbb{R}^3$ whose closed boundary surface $\partial\Omega$ has outward unit normal $\hat{n}$, and let $\vec{F}$ be C^1 on Ω and $\partial\Omega$. Then

ON THE FORMULA SHEET

$$\iint_{\partial\Omega} \vec{F} \cdot \hat{n} dS = \iiint_{\Omega} \nabla \cdot \vec{F} dV.$$

It holds only for *closed* surfaces (a sphere: no edge), and it often swaps a surface parameterisation for an easier triple integral. Since the left side is a flux, $\nabla \cdot \vec{F}$ is **flux per unit volume**: the net outflow from an infinitesimal sphere about a point, positive for a net outflow. The global flux

out of the surface is the sum of those local elements inside it. The 2D radial field (x, y) points away from the origin and grows outward; its divergence is $2 > 0$, a source (Lecture 11 p.10): $\nabla \cdot \vec{F} > 0$ means outflow.

A boundary may be several surfaces (Lecture 11 p.12): the solid half ball $x^2 + y^2 + z^2 \leq 1$, $z \geq 0$ is bounded by the hemisphere Σ_1 and the disk Σ_2 ($z = 0$), so its outward flux is $\iint_{\Sigma_1} + \iint_{\Sigma_2}$, two parameterisations – or one triple integral. An open surface must be closed off before the theorem applies (Problem Set 5 Q2): for the flow $\rho_0(kx, ky, kz)$ the side of the cylinder $x^2 + z^2 = b^2$, $|y| \leq \ell$, carries $4\pi\rho_0 kb^2\ell$, each end $y = \pm\ell$ another $\pi\rho_0 kb^2\ell$, and only the total, $6\pi\rho_0 kb^2\ell$, equals $\iiint 3\rho_0 k dV$. Symmetry often finishes the triple integral: $\nabla \cdot (x^2, y^2, z^2) = 2(x + y + z)$ is odd in each variable, so over the cube $|x|, |y|, |z| \leq 1$ it integrates to 0 (Problem Set 5 Q1, which redoes Problem Set 4's fluxes this way: the heat flux $-k\nabla T$ of $T = x^2 + y^2 + z^2$ out of the unit sphere is $-6k \cdot \frac{4}{3}\pi = -8\pi k$).

WORKED EXAMPLES --

LECTURE 11 PP.11, 13-14

Flux of $\vec{F} = (x, y, z)$ out of the sphere of radius R : $\nabla \cdot \vec{F} = 3$, so the flux is $3 \cdot \frac{4}{3}\pi R^3 = 4\pi R^3$. For any radial field $\vec{F} = f(r)\vec{r}$ through the sphere of radius b (Lecture 12 p.17, Problem Set 5 Q3), $\hat{n} = \vec{r}/r$ gives $\vec{F} \cdot \hat{n} = rf(r) = bf(b)$, constant, so the flux is $bf(b) \cdot 4\pi b^2 = 4\pi b^3 f(b)$; $f = 1$ is the case above.

The theorem needs $\vec{F}$ to be C^1 inside. For a point charge, $\vec{E} = q\vec{r}/r^3$, $\nabla \cdot \vec{E} = 0$ away from the origin, so the flux out of any closed surface not enclosing the charge is 0; a sphere about the charge contains the singularity, and the direct computation gives $4\pi q = Q/\epsilon_0$ (Problem Set 5 Q3, $q = Q/4\pi\epsilon_0$).

Outward flux of $\vec{F} = (xz, yz, xy)$ from the solid cone $z = 1 - \sqrt{x^2 + y^2}$ above the xy -plane (boundary: the cone's side and its base disk):

$\nabla \cdot \vec{F} = z + z + 0 = 2z$, and in cylindrical coordinates $0 \leq z \leq 1 - r$, $0 \leq r \leq 1$, or $0 \leq r \leq 1 - z$, $0 \leq z \leq 1$ (the lecture gives both orders and stops; it writes only “ $2\pi/3$?”). These notes' evaluation, which the other order confirms as $2\pi \int_0^1 z(1 - z)^2 dz = \pi/6$:

$$\begin{aligned} \iiint_{\Omega} 2z dV &= \int_0^{2\pi} \int_0^1 \int_0^{1-r} 2z r dz dr d\theta \\ &= 2\pi \int_0^1 r(1 - r)^2 dr = 2\pi \cdot \frac{1}{12} = \frac{\pi}{6}. \end{aligned}$$

Conservation laws (Lecture 11 pp.15-16). Two results turn the theorem into a differential equation. You may differentiate under the integral when the variables differ,

$$\frac{d}{dt} \iiint_{\Omega} f(\vec{x}, t) dV = \iiint_{\Omega} \frac{\partial f}{\partial t} dV;$$

and if $\iiint_{\Omega} f \, dV = 0$ for every Ω , with f continuous, then $f = 0$ (the du Bois-Reymond lemma; both are stated in one dimension and hold for triple integrals). Gauss's law – electric flux out of a closed surface is proportional to the enclosed charge – then gives Maxwell's first equation (Lecture 11 p.16), with charge density ρ in Ω :

$$\iint_{\partial\Omega} \vec{E} \cdot \hat{n} \, dS = \frac{1}{\epsilon_0} \iiint_{\Omega} \rho \, dV \Rightarrow \iiint_{\Omega} \left(\nabla \cdot \vec{E} - \frac{\rho}{\epsilon_0} \right) dV = 0 \quad \forall \Omega \Rightarrow \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}.$$

The continuity equation of fluid mechanics comes the same way; Lecture 11 p.17 leaves it to Möbius, which is graded, so it is not worked here.

12 Stokes' theorem and potentials

Week 7 in the outline. Source: Lecture 12 pp.1-10, 15-17; primer 5 pp.12-23; Problem Set 5 Q4-6.

Theorem (Stokes; Lecture 12 p.1). Let Σ be a surface with closed boundary curve $\partial\Sigma$, the two oriented by the right-hand rule, and $\vec{F}$ be C^1 on $\Sigma \cup \partial\Sigma$. Then

ON THE FORMULA SHEET

$$\iint_{\Sigma} (\nabla \times \vec{F}) \cdot \hat{n} \, dS = \oint_{\partial\Sigma} \vec{F} \cdot d\vec{x}.$$

Σ must be *open* to have a boundary curve. Right-hand rule: fingers along the direction of $\partial\Sigma$, thumb along $\hat{n}$ – in $\mathbb{R}^3$ “counterclockwise” depends on where you stand. For Σ a plane region D and $\vec{F} = (F_1, F_2, 0)$, $\hat{n} = \hat{k}$ and $(\nabla \times \vec{F}) \cdot \hat{k} = \partial F_2 / \partial x - \partial F_1 / \partial y$: Stokes is Green. The component of the curl along $\hat{n}$ is the **circulation per unit area** in the plane perpendicular to $\hat{n}$, so the global circulation round $\partial\Sigma$ is the sum of the local circulation elements on Σ .

WORKED EXAMPLE -- A
CIRCLE IN A PLANE
(LECTURE 12 PP.2-3)

$\oint_C \vec{F} \cdot d\vec{x}$ for $\vec{F} = (y, 2z, 3x)$, C any circle of radius R in the plane $x + 2y - 2z = 1$. Σ is not given; the flat disk C bounds is the best choice, because its $\hat{n}$, the plane's unit normal, is constant:

$$\nabla \times \vec{F} = (0 - 2, 0 - 3, 0 - 1) = (-2, -3, -1), \quad \hat{n} = \frac{1}{3}(1, 2, -2),$$

$$(\nabla \times \vec{F}) \cdot \hat{n} = \frac{1}{3}(-2 - 6 + 2) = -2,$$

so $\oint_C \vec{F} \cdot d\vec{x} = -2 \cdot \pi R^2 = -2\pi R^2$ for C oriented with that $\hat{n}$ by the right-hand rule, $+2\pi R^2$ the other way (the page fixes neither). The curl is constant, so only the circle's area matters, not where it sits in the plane.

Conservative, irrotational, circulation-free (Lecture 12 p.4). For $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ these are equivalent: (1) $\vec{F} = \nabla\phi$ for some ϕ ; (2) $\nabla \times \vec{F} = \vec{0}$; (3) $\oint_C \vec{F} \cdot d\vec{x} = 0$ for every closed curve C . (1) $\Rightarrow$ (2) is the curl-of-a-gradient identity; (2) $\Rightarrow$ (3) is Stokes on any surface spanning C ; (3) $\Rightarrow$ (1) is beyond the course. In $\mathbb{R}^2$ the curl test was $\partial F_2/\partial x = \partial F_1/\partial y$.

Vector potential (Lecture 12 pp.5-7). The divergence-of-a-curl identity has a converse: if $\nabla \cdot \vec{F} = 0$, there is a field $\vec{A}$ with $\vec{F} = \nabla \times \vec{A}$, the **vector potential**. By Stokes, a flux of such an $\vec{F}$ through an open surface becomes $\oint_{\partial\Sigma} \vec{A} \cdot d\vec{x}$ (you trade a surface integral for a line integral, but must find all three A_i), so it depends only on the boundary curve:

ON THE FORMULA SHEET

$$\nabla \cdot \vec{F} = 0, \quad \partial\Sigma_1 = \partial\Sigma_2 \implies \iint_{\Sigma_1} \vec{F} \cdot \hat{n} dS = \iint_{\Sigma_2} \vec{F} \cdot \hat{n} dS.$$

Lecture 12 p.7 proves it by applying Stokes to each surface. Problem Set 5 Q5 shows it: for $\vec{F} = (y - z, -x - z, x + y)$, $\nabla \times \vec{F} = (2, -2, -2)$ has flux -18π upward through both the paraboloid $z = 9 - x^2 - y^2$, $z \geq 0$ (by the circulation round $x^2 + y^2 = 9$) and the flat disk $(-2 \cdot 9\pi)$. When a curve lies in a plane, the flat piece of that plane is the easy surface: the triangle $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ lies in $x + y + z = 1$, and for $\vec{F} = (xy, yz, zx)$, $(\nabla \times \vec{F}) \cdot \hat{n} = -(x + y + z)/\sqrt{3} = -1/\sqrt{3}$ there, times the area $\sqrt{3}/2$: circulation $-\frac{1}{2}$ (Problem Set 5 Q4).

THE TWO POTENTIALS SIDE BY SIDE (LECTURE 12 P.9)

$\nabla \times \vec{F} = \vec{0}$	$\nabla \cdot \vec{F} = 0$
ϕ exists with $\vec{F} = \nabla\phi$ (scalar potential)	$\vec{A}$ exists with $\vec{F} = \nabla \times \vec{A}$ (vector potential)
$\int_C \vec{F} \cdot d\vec{x}$ is independent of the path from $\vec{a}$ to $\vec{b}$	$\iint_{\Sigma} \vec{F} \cdot \hat{n} dS$ is independent of the surface spanning $\partial\Sigma$

**WORKED EXAMPLE --
SWAPPING THE
SURFACE (LECTURE 12
P.8)**

Outward flux of $\vec{F} = (-x, y, 1)$ through the hemisphere $z = \sqrt{1 - x^2 - y^2}$. Parameterising is tedious, but $\nabla \cdot \vec{F} = -1 + 1 + 0 = 0$, so any surface with the same boundary circle $x^2 + y^2 = 1$, $z = 0$, oriented to match, gives the same flux. The unit disk $z = 0$ with $\hat{n} = \hat{k}$ (upward, to match the dome's outward normal round the circle; the lecture does not say why) has $\vec{F} \cdot \hat{k} = 1$, so the flux is its area, π .

Finding $\vec{A}$ (Lecture 12 pp.15-16, Problem Set 5 Q6). $\vec{F} = \nabla \times \vec{A}$ is three equations, $F_1 = \partial_y A_3 - \partial_z A_2$, $F_2 = \partial_z A_1 - \partial_x A_3$, $F_3 = \partial_x A_2 - \partial_y A_1$. $\vec{A}$ is not unique, so set a term to zero to start.

WORKED EXAMPLE --

LECTURE 12 P.16

$\vec{F} = (x, y, -2z)$, $\nabla \cdot \vec{F} = 0$. Assume $\partial_z A_2 = 0$: then $A_3 = xy$;
 $\partial_z A_1 = y + y = 2y$, so $A_1 = 2yz$; the third equation needs $\partial_x A_2 = 0$, so
 $A_2 = 0$.

$$\vec{A} = (2yz, 0, xy), \quad \nabla \times \vec{A} = (x, 2y - y, -2z) = \vec{F}.$$

Another choice of zero term gives another $\vec{A}$ with the same curl.

Problem Set 5 Q6 uses this to avoid a surface: $\vec{F} = (xz, -yz, 1 + y^2)$ has zero divergence, $\vec{A} = (0, x(1 + y^2), xyz)$ works (start from $A_3 = xyz$ for F_1 and F_2), and the flux up through the cap $z = \cos^{-1}(x^2 + y^2)$ is $\oint \vec{A} \cdot d\vec{x}$ round the unit circle, $\int_0^{2\pi} \cos^2 t (1 + \sin^2 t) dt = 5\pi/4$ – the same as through the unit disk.

Integral-of-a-derivative theorems (Lecture 12 p.10). Each integrates a derivative over a region and gets the field on the boundary:

$$\int_a^b f' dx = f(b) - f(a), \quad \int_C \nabla \phi \cdot d\vec{x} = \phi(\vec{b}) - \phi(\vec{a}),$$

$$\iint_{\Sigma} (\nabla \times \vec{F}) \cdot \hat{n} dS = \oint_{\partial\Sigma} \vec{F} \cdot d\vec{x}, \quad \iiint_{\Omega} \nabla \cdot \vec{F} dV = \iint_{\partial\Omega} \vec{F} \cdot \hat{n} dS.$$

13 Maxwell's equations

Week 7 in the outline. Source: Lecture 12 pp.12-14; primer 5 pp.25-27; Problem Set 5 Q7-9.

Each law of electromagnetism is a statement about a flux or a circulation; the Divergence Theorem turns a closed-surface flux law into a divergence equation, and Stokes' theorem turns a circulation law into a curl equation. The derivations (Problem Set 5 Q7) all run the same way: write the law as integrals over an arbitrary surface or region, apply the theorem to the side that is not yet a flux or volume integral, move any ∂_t inside, collect under one integral equal to zero, and apply the du Bois-Reymond lemma:

MAXWELL'S EQUATIONS, AS LECTURE 12 P.12 AND PRIMER 5 P.25 WRITE THEM

EQUATION	FROM
$\nabla \cdot \vec{E} = \rho/\epsilon_0$	Gauss's law: electric flux out of a closed surface is proportional to the enclosed charge (Divergence)
$\nabla \times \vec{E} = -\partial \vec{B}/\partial t$	Faraday: circulation of $\vec{E}$ round a boundary is minus the rate of change of magnetic flux through the surface (Stokes)
$\nabla \cdot \vec{B} = 0$	Gauss for magnetism: magnetic flux through any closed surface is zero (Divergence)
$\nabla \times \vec{B} = \mu_0 \epsilon_0 \partial \vec{E}/\partial t + \mu_0 \vec{J}$	Ampère: circulation of $\vec{B}$ follows the rate of change of electric flux and the current density (Stokes)

They are four coupled PDEs: $\vec{E}$ and $\vec{B}$ cannot be solved for separately. With no time dependence they decouple into electrostatics, $\nabla \cdot \vec{E} = \rho/\epsilon_0$, $\nabla \times \vec{E} = \vec{0}$, and magnetostatics, $\nabla \cdot \vec{B} = 0$, $\nabla \times \vec{B} = \mu_0 \vec{J}$ (Lecture 12 p.13). Each has a potential:

- **Electrostatics:** $\nabla \times \vec{E} = \vec{0}$, so $\vec{E} = \nabla \phi$ and Gauss gives **Poisson's equation** $\nabla^2 \phi = \rho/\epsilon_0$ (Laplace's when $\rho = 0$). Physics writes $\vec{E} = -\nabla V$, so $\nabla^2 V = -\rho/\epsilon_0$: the same law with $V = -\phi$.
- **Magnetostatics:** $\nabla \cdot \vec{B} = 0$, so $\vec{B} = \nabla \times \vec{A}$ and $\nabla \times \vec{B} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$. $\vec{A}$ is not unique, so choose $\nabla \cdot \vec{A} = 0$: $\nabla^2 \vec{A} = -\mu_0 \vec{J}$, a vector Poisson equation.

With no charges and no currents ($\rho = 0$, $\vec{J} = \vec{0}$) Lecture 12 p.14 derives the wave equation: take ∂_t of the fourth equation, swap the derivatives, substitute the second, and use $\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$ with $\nabla \cdot \vec{E} = 0$:

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$$\frac{\partial^2 \vec{E}}{\partial t^2} - c^2 \nabla^2 \vec{E} = \vec{0}, \quad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

With given sources ρ and $\vec{J}$ the same steps (curl of Faraday, then Gauss and Ampère substituted) give **inhomogeneous** wave equations (Problem Set 5 Q8), and the curl of Ampère with $\nabla \cdot \vec{B} = 0$ does the same for $\vec{B}$:

$$\frac{\partial^2 \vec{E}}{\partial t^2} - c^2 \nabla^2 \vec{E} = -\frac{1}{\epsilon_0} \left(c^2 \nabla \rho + \frac{\partial \vec{J}}{\partial t} \right), \quad \frac{\partial^2 \vec{B}}{\partial t^2} - c^2 \nabla^2 \vec{B} = \frac{1}{\epsilon_0} \nabla \times \vec{J}.$$

WORKED EXAMPLE -- A $\phi = -\rho_0 R^3 / (3\epsilon_0 r)$ for $r \geq R$ and $\phi = \rho_0 r^2 / (6\epsilon_0) - \rho_0 R^2 / (2\epsilon_0)$ for $r < R$.
UNIFORMLY CHARGED With $\nabla\phi = \phi'(r) \vec{r}/r$, $\vec{E} = \rho_0 R^3 \vec{r} / (3\epsilon_0 r^3)$ outside and $\rho_0 \vec{r} / (3\epsilon_0)$ inside;
BALL (PROBLEM SET 5 $\rho = \epsilon_0 \nabla \cdot \vec{E}$ is 0 outside and ρ_0 inside. The flux out of the sphere $r = 2R$
Q9) counts only the ball where $\nabla \cdot \vec{E} \neq 0$: $\frac{\rho_0}{\epsilon_0} \cdot \frac{4}{3} \pi R^3$.

(The course overview lesson writes the same laws with $\vec{D} = \epsilon_0 \vec{E}$ and $\vec{H} = \vec{B} / \mu_0$ in vacuum:
 $\nabla \cdot \vec{D} = \rho$, $\nabla \times \vec{H} = \vec{J} + \partial \vec{D} / \partial t$.)

14 Complex differentiability, contours and residues

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$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}, \quad h \in \mathbb{C}, \quad u_x = v_y, \quad u_y = -v_x.$$

The limit must agree from every direction h , which gives the Cauchy-Riemann equations. Differentiable on an open set means analytic; u, v are then harmonic conjugates, and where $f'(z) \neq 0$ the map is conformal (angle-preserving).

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$$\int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt, \quad f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz.$$

Cauchy-Goursat: zero for f analytic on and inside a simple closed contour, so a contour shrinks onto small circles around the singularities; Cauchy's formula gives $f(z_0)$.

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$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad \text{Res}(f, z_0) = a_{-1}, \quad \oint_C f(z) dz = 2\pi i \sum_k \text{Res}(f, z_k).$$

The Laurent series's negative part classifies the singularity (none: removable; finitely many, to order m : pole of order m ; infinitely many: essential). Only the a_{-1} term survives a loop, so a contour integral is $2\pi i$ times the enclosed residues. Simple pole: $\text{Res} = \lim_{z \rightarrow z_0} (z - z_0) f(z)$.